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THE σ-HULL – THE HULL WHERE FRACTALS LIVE
CALCULATING A HULL BOUNDED BY LOG SPIRALS TO SOLVE
THE INVERSE IFS-PROBLEM BY THE DETECTED ORBITS

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Global IFS seem to be suited best for compressed encoding of natural objects which are in most cases self affine even if not always exactly. Since affine transformations - the IFS-Codes - resp. the union of all their orbits generate an object (an IFS-Attractor), the detection of a non minimal set of these orbits solves the inverse IFS-Problem by calculating a superset of IFS-Codes, which has to be minimized.
Here a method is presented how these orbits (in particular those on the object boundary) can be calculated. Therefore a generalized convex hull - the σ-Hull - is defined. This fractal hull is bounded by log spirals, that curves formed by the orbits. It is shown that log spirals can be represented by a continuos function of powers of affine maps and that by using this "spiral equivalent" the generating transformations of the orbits by which an IFS-object is bound, can be calculated in the x/y-plane.
Further more this representation can be used for the classification of the detected orbits, necessary to calculate the IFS-Codes of a minimal IFS from their generating transformations, subsequently.

1 Introduction

1.1 Problem Specification - Objects to be encoded

The idea is to globally encode[1] objects of an image appearing in the nature and therefore often self similar to a high degree but difficult to encode or to compress by a mathematical formula. This *global IFS-Approach*[2] - representing an object by the union of affine contractive transformed copies (subobjects generated by the IFS-Codes) of the object itself - seems to be used best to solve the inverse (global) IFS-Problem.

Thus the *Collage Theorem* which can be used to generate whole objects of an image is given: A set $\{\omega_i: R^2 \to R^2 \mid \omega_i$ affine, contraction$\}$ of complete Euclidean space (R^2, d) is an *Iterated Function Systems (IFS)* with contractivity factor s, for $i \in N$, the distance d, $0 \le s < 1$ and $s \cdot h(B, C) \ge h(W(B), W(C))$, if for the Hutchinson operator W^a holds

$$W(A) = \bigcup_{\omega_i \in W} \bigcup_{x \in A} \omega_i(x) \text{ and } A = W(A).$$

The compact set A is called an *attractor* of W.
The *Collage Theorem* (CT) for attractors close to given compact sets:

$$h(L, W(L)) \le \varepsilon \Rightarrow h(L, A) \le {}^{\varepsilon}/_{(1-s)}$$

where h is called the *Hausdorff Distance*

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[a] W: $\mathcal{H}(R^2) \to \mathcal{H}(R^2)$ with W(A) = {W(a) | a∈A}, A∈$\mathcal{H}(R^2)$ is a contractive map of ($\mathcal{H}(R^2)$,h) the space of nonempty compact subsets of R^2 where h(C,B) = max{d(C,B), d(C,A)} and C, B∈$\mathcal{H}(R^2)$.

$L \subseteq \mathbb{R}^2$ is a given compact set of $(\mathbb{R}^2, d)$ and $\varepsilon \geq 0$. For $\varepsilon = 0$ $(L = A)$ the CT for exactly self affine attractors is obtained. In principle the CT can be applied to any object if sufficient many IFS-Codes ω_i are used. However, to compress an object the number of parts ω_i (L) (the subobjects) have to be small for a small ε.

For the discrete pixel space the *discrete CT*[3] is defined as a set $\{\omega_m \colon \mathbb{P}^2 \to \mathbb{P}^2 \mid \omega_m$ affine, contraction$\}$ of the pixel space $(\mathbb{P}^2, d)$ - $P = \{0,...,R-1\} \subset \mathbb{N}$ - is a *Discrete Iterated Function Systems (DIFS)* with contractivity factor s, for $m \in P$, R^2 the number of image screen pixels, the block distance d, $0 \leq s < 1$ and $s \cdot h(B,C) \geq h(W(B), W(C))$, if for the Hutchinson operator W: $\mathcal{H}(\mathbb{P}^2) \to \mathcal{H}(\mathbb{P}^2)$ holds

$$W(A) = \bigcup_{m \in P} \bigcup_{x \in A} \omega_m(x) \text{ and } A = W(A).$$

The set A is called an discrete *attractor* of W. The discrete attractor $A \in \mathcal{H}(\mathbb{P}^2)$ is not unique, but there always exist a unique maximal attractor $A_{max} \in \mathcal{H}(\mathbb{P}^2)$ with $A = W(A) \Rightarrow A \subseteq A_{max}$, which contains all discrete attractors.

The *Chaos Game* generates an attractor A by the iterated random application of the different ω_m to one image point i.e., by the union of orbits formed by all combinations of ω_m - the basic principle of the presented solution.

In contrast to global IFS, an *encoding* based on *Partitioned IFS* (PIFS)[4] divides the image into larger and smaller segments (polygons). The smaller target segments are interpreted as affine contractive copies of the larger ones and then the CT is applied. Since only IFS-Properties of affine relationships constrained to the segment form can be exploited, all of the many image parts are only approximated by an IFS-encoding.

1.2 The Solution - How to encode global IFS-Objects

Calculating boundaries of an object

In general the boundary of an object contains the most information how an object is built. Points inside an object normally cannot be related, because they are difficult to distinguish. The common method to determine the boundary of an object - especially if it is defined in a discrete (pixel) space - is to form the hull of an object.

We first can try it by the convex hull $\mathcal{H}A)$, which is defined as "Intersection of half planes containing a point set (the attractor) A". It soon appears that a hull of such a kind is not usable for IFS-Attractors, because only the boundary of convex objects - what an IFS certainly is not - can be determined.

Second we can try a more general hull - the α-Hull[5] $\mathcal{H}_\alpha$ - defined as follows :
The α-Hull $\mathcal{H}_\alpha(A)$ (see *Fig. 1*) is defined as the intersection of all generalized disks B(c, r) resp. B(c, -r) of non negative radius $r = 1/\alpha$ and the centre c containing the attractor A, where B(c, -r) is the complement of the disk B(c,r).[6]

The boundary of an α-Hull is formed by α-Extremes (see *Fig. 1*):
A point $s \in A$ is termed α-*Extreme* (corner of $\mathcal{H}_\alpha(A)$) in A, if there exists a B(c, $1/\alpha$) such that s lies on its boundary and contains all other points of A.[6]

Now it is possible to analyse concave objects and even objects concave with different curvature using different radii of the generalized disks. But we need a relationship of boundary points (extremes) which provides more information about the object. Using the

α-Hull we get only a relationship of points laying on the same edge of a disk of different radius forming the α-Hull. But the boundary points of an IFS-Attractor do not lay on circles. They form orbits[b] in the discrete space which are given as subset of W^+ [c] for an IFS W (as it is easily to observe in the application of the Chaos Game). These orbits lie on logarithmic spirals[1], what will be shown in this paper. So the best suited hull to determine an IFS-Attractor seems to be the σ-Hull which is bounded by log spirals and will be detailly defined and described in the main part of this paper.

Calculating a minimal IFS

Once found the orbits on the boundaries of different σ-Hulls they have to be classified to calculate the IFS-Codes:

> Classes C_i = {orbits which can be affinely mapped onto eachother} are formed.
> The potential IFS-Codes ϖ_i are the generating transformations of the largest orbits[d] of each class.

Then the final IFS-Codes ω_i are the generating transformations of the largest orbits within each C_i which can be mapped onto eachother by (have an inter orbit transformation in) $\{\omega_i\}^+$.

Thus the final and also minimal IFS can be calculated by the following iterative process

(a) which restrains the classes to those orbits which can be mapped onto each other only by ϖ_i detected so far
  $\Rightarrow$ more and more classes are generated (and the set $\{\varpi_i\}$ is increased)
(b) which enlarges the classes to those orbits which can now be mapped onto each other only by ϖ_i detected so far
  $\Rightarrow$ fewer and fewer classes are generated (and the set $\{\varpi_i\}$ is decreased)

This process is repeated until no more (in (a)) resp. no fewer (in (b)) classes are generated. Then the generating transformations of the largest orbits of each remaining class form a minimal set of the IFS-Codes of the object.

Since a σ-Hull is only useful to find the orbits on the boundary of an object, orbits of inner subobjects (having no boundary in common with the object) cannot be found in the first step. But if you form the difference between the object A_0 and the point set generated by the IFS-Codes in the first step one will get a difference object A_1 in the second step. Those affine transformations mapping boundary orbits (detected by different σ-Hulls) of the object A_0 to those of the difference object A_1 are the IFS-Codes for the inner subobjects now lying outside. The second step has to be repeated for the object A_i ($i \in \mathbb{N}$) as long as the difference object A_{i+1} is not empty. Adjusting the obtained results and eliminating possible deviations by comparing the inverse IFS-mapping of the hull of inner subobjects with the object hull, also the inverse IFS-Problem for overlapping subobjects can be solved.

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[b] An *orbit O* is a point sequence $o_n = \phi^{(n)}(x)$ $\forall n \in \mathbb{N}$ where $\phi^{(n)} = \prod_n \phi$, ϕ affine and $x \in \mathbb{R}^2$.

The point sequence (orbit) $\phi^{(n)}(p)$ converges for a contractive affine maps ϕ to the *fixed point* f of ϕ

[c] $W^* = \{ \prod_{n \in \mathbb{N}} \omega_n \mid \omega_n \in W \cup \{\epsilon\}$ and $W^+ = W^* \setminus \{\epsilon\}$ with $W = \{\varpi_i \mid \varpi_i$ affine$\}$ and ϵ the *Identity Map*.

[d] The *largest orbits* of a subobject $\rho(A)$ ($\rho \in W \cup \{\epsilon\}$) are those orbits $O = \{o_0, ..., o_n\}$ with a minimal contraction, where $\phi^{(n)}(o_0) \in A$ for $n \geq 0$ and $\phi^{(n)}(o_0) \notin A$ for $n < 0$ and $\phi \in W^+$.

2   Definition and Application of the σ-Hull

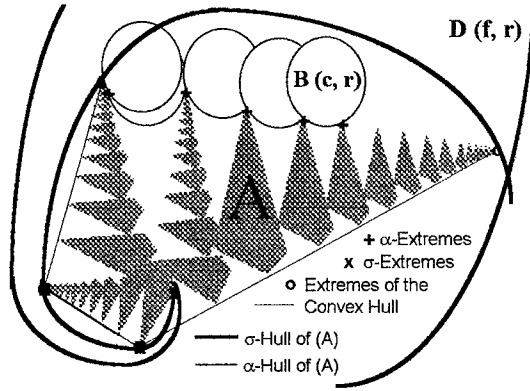### 2.1   *The σ-Hull (fractal hull) in* $\mathbb{R} \times \varphi$

Figure 1. The convex Hull, the α- and σ-Hull of an Object

We first introduce the notion of generalized spiral disks. Let D*(f, r) denote a spiral disk defined by the point set $\bigcup_i^{\infty} p_i$ with fixed point (centre) f, the non negative radius $r = Re^{k\varphi}$ where $p_i \in \mathbb{R} \times \varphi$ and $d(f, p_i) \leq r$ and $0 \leq \varphi \leq 2m\pi$, $m \in \mathbb{N}$.

A *generalized spiral disk*

$$D'(f, r) \equiv \begin{cases} D*(f, -r)^C & r < 0 \\ D*(f, r) & r \geq 0 \end{cases} \tag{1}$$

where X^C is the complement of X.

Since D'(f, r) is not solely bounded by a log spiral (but also by a part of the $\varphi = 0$ axis) a generalized spiral disc segment has to be constructed bounded only by equally curved log spirals having the same fixed point.

Thus a *generalized spiral disk segment* is given by *(Fig. 1)*:

$$D(f, r) = D'(f, Re^{k\varphi}) \cap D'(f, -Re^{k(\pi/4+\varphi)}) \cap D'(f, Re^{k(\pi/2-\varphi)}) \tag{2}$$

Note: For negative k one will get a spiral disc segment reflected by the $\varphi = 0$ axis !

$C_\sigma(S)$ is the *set of fixed points* of spiral disc segments of radius $1/\sigma$ that have S as subset

$$C_\sigma(S) = \{x \in \mathbb{R} \times \varphi \mid S \subseteq \overline{D(x, 1/\sigma)}\} \tag{3}$$

where $\overline{X}$ is the set closure of X.

Let X be an arbitrary (finite or connected set) in $\mathbb{R} \times \varphi$ then the *intersection of all generalized spiral disc segments* with varying fixed point $x \in X$ and a fixed radius $1/\sigma$

$$\text{is denoted by } M_\sigma(X) \equiv \begin{cases} R x \varphi & X = \varnothing \\ \bigcap_{x \in X} D(x, {}^1/\sigma) & X \neq \varnothing \end{cases} \tag{4}$$

where $\varnothing$ is the empty set.

The σ-*Hull* of S is the intersection of all generalized spiral disk segments of radius $^1/_\sigma$ that contain all points of S. Thus the σ-*Hull (fractal hull)* is given by

$$\mathcal{H}_\sigma(S) \equiv M_\sigma(C_\sigma(S)) \tag{5}$$

A point $s \in S$ is termed σ-*Extreme* (corner of $\mathcal{H}_\sigma(S)$) in S, if there exists a $D(c, {}^1/_\sigma)$ such that s lies on its boundary and contains all other points of S (see *Fig. 1*).

Thus the set of σ-*Extremes* is defined by

$$E_\sigma(S) \equiv \{s \in S \mid \exists c \in C_\sigma(S): s \in \partial(c, {}^1/_\sigma)\} \tag{6}$$

where ∂X is the boundary of S.

As in the limit case of infinite radius (i.e. $^1/_k$ close to 0) resp. of radius $= R$ ($k = 0$) the generalized spiral disc segment becomes a half plane resp. a circle of radius R, this definition includes the definition of the convex Hull $\mathcal{H}(S)$ resp. of the α-Hull $\mathcal{H}_\alpha(S)$ as well.

2.2   *Defining the σ-Hull in $\mathbb{R}^2$*

To represent a log spiral $r = Re^{k\varphi}$ defined in the polar plane ($\mathbb{R} \times \varphi$) an equivalent for the x/y-plane ($\mathbb{R} \times \mathbb{R}$) has to be found. The best suited equivalent seems to be one formulated in terms of affine transformations. In addition, this representation will relate log spirals and orbits (lying on the log spirals) which union forms an IFS and where the generating transformations of the largest of these orbits are the IFS-Codes.

Theorem 1 - *Log spiral equivalent* - A log spiral $r = Re^{k\varphi}$ can be represented by a continuos function of powers of an affine transformation ω.

$$Re^{k\varphi} \equiv \omega^n = R_0(\alpha)\, S(c_x{}^n, c_y{}^n)\, R_0(-\alpha)\, R_0(n(-\beta)),$$

where $n \in \mathbb{R}$, $\alpha \in [-\pi, \pi)$, $\beta \in [-\pi, \pi)$ and $\beta \neq 0$

Lemma 1 Each affine transformation can be represented by a symmetric matrix and a rotation matrix.

$$\forall \omega = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2x2} \ \exists \beta \in [-\pi, \pi) \ \exists \psi = \begin{pmatrix} r & s \\ s & t \end{pmatrix} \in \mathbb{R}^{2x2} :$$

$$\omega = \psi R_0(-\beta) = \begin{pmatrix} r\cos\beta + s\sin\beta & s\cos\beta - r\sin\beta \\ s\cos\beta + t\sin\beta & t\cos\beta - s\sin\beta \end{pmatrix} \text{ with } \tan\beta = \frac{b-c}{a+d}$$

$$r = \frac{a^2 + b^2 + ad - bc}{\sqrt{(a+d)^2 + (c-b)^2}}, \quad t = \frac{c^2 + d^2 + ad - bc}{\sqrt{(a+d)^2 + (c-b)^2}} \quad \text{and} \quad s = \frac{ac - bd}{\sqrt{(a+d)^2 + (c-b)^2}}$$

Proof of Lemma 1: in [3]

Lemma 2 Each symmetric matrix can be represented by a rotation α, a scaling matrix and a rotation $-\alpha$.

$$\forall \psi = \begin{pmatrix} r & s \\ s & t \end{pmatrix} \in \mathbb{R}^{2\times2} \ \exists \ \alpha \in [-\pi, \pi) \ \exists \ c_x, c_y \in \mathbb{R} : \psi = R_0(\alpha)\, S(c_x, c_y)\, R_0(-\alpha)$$

with the eigenvalues $c_x = \dfrac{r + t + \sqrt{(r-t)^2 + 4s^2}}{2}$, $\qquad c_y = \dfrac{r + t - \sqrt{(r-t)^2 + 4s^2}}{2}$

and $\alpha = \arctan\dfrac{c_x - r}{s}$ the angle of the eigenvector $\vec{e}_x = \begin{pmatrix} c_x - r \\ s \end{pmatrix}$

Proof of Lemma 2: in [3]

Lemma 3 Each affine transformation ω maps a point $\overline{x} \in Re^{k\varphi}$ again to a point $\overline{x}_1 \in Re^{k\varphi}$.

$$\forall \omega \in \mathbb{R}^{2\times2} \text{ and } \overline{x} \in Re^{k\varphi} : \omega\overline{x} = \overline{x}_1 \text{ and } \overline{x}_1 \in Re^{k\varphi}$$

Proof of Lemma 3:

$r = Re^{k\varphi}$ resp. $\sqrt{x^2 + y^2} = Re^{k\arctan\frac{y}{x}}$

$x = r\cos\varphi = Re^{k\varphi}\cos\varphi$ and $y = r\sin\varphi = Re^{k\varphi}\sin\varphi$

$$\omega\overline{x} = \omega\begin{pmatrix} x \\ y \end{pmatrix} = \omega\begin{pmatrix} Re^{k\varphi}\cos\varphi \\ Re^{k\varphi}\sin\varphi \end{pmatrix} = Re^{k\varphi}\begin{pmatrix} a\cos\varphi + b\sin\varphi \\ c\cos\varphi + d\sin\varphi \end{pmatrix} = \overline{x}_1$$

$\varphi = \arctan\dfrac{y}{x} = \arctan\dfrac{a\cos\varphi + b\sin\varphi}{c\cos\varphi + d\sin\varphi}$ and

$r = \sqrt{x^2 + y^2} = Re^{k\varphi}\sqrt{(a\cos\varphi + b\sin\varphi)^2 + (c\cos\varphi + d\sin\varphi)^2}$

$Re^{k\varphi}\sqrt{(a\cos\varphi + b\sin\varphi)^2 + (c\cos\varphi + d\sin\varphi)^2} = Re^{k\varphi}e^{k\arctan\frac{c\cos\varphi + d\sin\varphi}{a\cos\varphi + b\sin\varphi}} \ \Big| \ :e^{k\varphi}$

$\underbrace{\sqrt{R^2(a\cos\varphi + b\sin\varphi)^2 + R^2(c\cos\varphi + d\sin\varphi)^2}}_{} = Re^{k\arctan\frac{c\cos\varphi + d\sin\varphi}{a\cos\varphi + b\sin\varphi}}$

$\qquad\qquad x^2 \qquad\qquad\qquad\qquad y^2 \qquad\qquad\qquad \arctan\dfrac{Ry}{Rx} = \arctan\dfrac{y}{x}$

Lemma 4 A power $n \in G$ of an affine transformation maps $\overline{x} \in Re^{k\varphi}$ again to $\overline{x}_1 \in Re^{k\varphi}$.

$$\forall \omega^n \in \mathbb{R}^{2\times2} \text{ and } \overline{x} \in Re^{k\varphi} : \omega^n\overline{x} = \overline{x}_1 \text{ and } \overline{x}_1 \in Re^{k\varphi} \text{ where } n \in G$$

Proof of Lemma 4:

Lemma 3 holds for each point on the log spiral $Re^{k\varphi}$

I.e. also for $\omega\overline{x}$ and for $\omega^2\overline{x}$ and so on ... until $\omega^n\overline{x}$ resp. also for $\omega^{-1}, \omega^{-2}, ..., \omega^{-n}$

Corollary 1 Each power of an affine transformation can be represented as follows.

$$\omega^n = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^n = R_0(\alpha)S(c_x{}^n, c_y{}^n)R_0(-\alpha)R_0(n(-\beta))$$

$$= \begin{pmatrix} c_x^n\cos^2\alpha + c_y^n\sin^2\alpha & (c_x^n - c_y^n)\cos\alpha\sin\alpha \\ (c_x^n - c_y^n)\cos\alpha\sin\alpha & c_y^n\cos^2\alpha + c_x^n\sin^2\alpha \end{pmatrix} R_0(n(-\beta))$$

Proof of Corollary 1:

Because of Lemma 4, Lemma 1 can be also defined for $n \in G$:

$$\omega^n = \psi^n R_0(-\beta)^n \text{ where } \psi = \begin{pmatrix} r & s \\ s & t \end{pmatrix}$$

$$\Rightarrow \omega^n = (\psi R_0(-\beta))^n = \psi^n R_0(-\beta)^n = \psi^n R_0(n(-\beta)) =$$
$$R_0(\alpha)\, S(c_x,\, c_y)\, R_0(-\alpha)R_0(\alpha)\, S(c_x,\, c_y)\, R_0(-\alpha)R_0(\alpha) \dots S(c_x,\, c_y)\, R_0(-\alpha)R_0(n(-\beta)) =$$
$$R_0(\alpha)\, S(c_x,\, c_y)\, R_0(0)\, S(c_x,\, c_y)\, R_0(0) \dots S(c_x,\, c_y)\, R_0(-\alpha)R_0(n(-\beta)) =$$
$$R_0(\alpha)\, S(c_x,\, c_y)^n\, R_0(-\alpha)R_0(n(\alpha+\beta)) = R_0(\alpha)\, S(c_x^{\,n},\, c_y^{\,n})\, R_0(-\alpha)\, R_0(n(-\beta))$$

Corollary 2 Lemma 4 can also be defined for powers $n \in R$ instead of $n \in G$.

$$\forall \omega^n \in R^{2 \times 2} \text{ and } \overline{x} \in Re^{k\phi} : \omega^n \overline{x} = \overline{x}_1 \text{ and } \overline{x}_1 \in Re^{k\phi} \text{ where } n \in R$$

Proof of Corollary 2:

Lemma 4 also holds for $n \in R$, if ω^n is represented according to Corollary 1, since powers $n \in R$ not applicable to ω are now defined for c_x and c_y (i.e. n is no more restricted to G)

Summary (Proof of Theorem 1)

Lemma 4 - using Lemma 3 - shows that ω^n ($n \in G$) generates point sequences on log spirals (the so-called orbits). Corollary 2 points out that for $n \in R$ the orbits of ω^n form log spirals. Corollary 1 defines - by means of Lemma 1 and Lemma 2 - a representation of $\omega^n = R_0(\alpha)\, S(c_x^{\,n},\, c_y^{\,n})\, R_0(-\alpha)\, R_0(n(-\beta))$ which can be used to calculate ω^n in $R \times R$ for $n \in R$, what concludes the chain of the proof for Theorem 1.

Definition of the σ-Hull in R^2

Let $D^*(f, \omega^n)$ be a spiral disk $\bigcup\limits_{i}^{\infty} p_i$ with fixed point f, bounded by ω^n with $0 \le n \le 2m\pi/\beta$,

$m \in N$ and radius $|\omega^n(s-f)|$, where $p_i \in R \times R$ and $d(f, p_i) \le |\omega^n(s-f)|$, $s \in R \times R$.

$$\text{Then } D'(f, \omega^n) \equiv \begin{cases} D^*(f, \omega^{-n})^C & n < 0 \\ D^*(f, \omega^n) & n \ge 0 \end{cases} \tag{7}$$

denotes a generalized spiral disk in R^2.

The generalized spiral disc segments described by means of log spirals in $R \times \phi$ as in (2) can now be defined using a continuos function of powers of affine transformations in R^2:

$$D(f, \omega^n) = D'(f, \omega^n) \cap D'(f, R_0(\pi/4)\omega^{-n}) \cap D'(f, R_0(\pi/2)S(-1,1)\omega^n) \tag{8}$$

where $\omega^n = R_0(\alpha)S(c_x^{\,n},\, c_y^{\,n})R_0(-\alpha)R_0(n(-\beta))$ and $0 \le c_x < 1$, $0 \le c_y < 1$, $-\pi \le \alpha < \pi$, $-\pi \le \beta < \pi$

Note: for negative α and β one will get a spiral disk segment reflected by the x-axis

Which - by adapting the definition (3) of the set of fixed points $C_\sigma(S)$ to $R \times R$ - leads to the definition of the intersection of generalized spiral disk segments $M_\sigma(X)$ resp. the σ-Hull in $R \times R$ instead of $R \times \phi$ in (4) resp. (5)

$$M_\sigma(X) \equiv \begin{cases} R \times R & X = \varnothing \\ \bigcap\limits_{x \in X} D(x,\, 1/|\omega^n(s-f)|) & X \ne \varnothing \end{cases} \text{ resp. } \mathcal{H}_\sigma(S) \equiv M_\sigma(C_\sigma(S)) \tag{9}$$

This leads to additional special cases for the σ-Hull defined in $\mathbb{R}^2$:

If $c_x = c_y$ in $R_0(\alpha)S(c_x{}^n, c_y{}^n)R_0(-\alpha)R_0(n(-\beta)$

$\Rightarrow D(f, \omega^n)$ is bounded by 2 straight lines forming an angle of $\pi/4$

If $c_x \neq c_y$ and $\alpha = 0$ in $R_0(\alpha)\ S(c_x{}^n, c_y{}^n)\ R_0(-\alpha)\ R_0(n(-\beta))$

$\Rightarrow D(f, \omega^n)$ is bounded by 3 exponential parabolas.

Both shapes are necessary to find all orbits (also those laying on non spiral like curves) by which an IFS is bound.

2.3 Calculating Orbits using the σ-Hull in $\mathbb{R}^2$

Calculation of Orbit Generating Transformations

Lemma 5 The curvature proportion ρ of $\omega^n \equiv Re^{k\varphi}$ is defined by a k independent of β

$$\forall \omega^n \equiv Re^{k\varphi} \text{ with } \rho = R/r : \quad k = \frac{\ln(c_x^2 \cos^2 \alpha + c_y^2 \sin^2 \alpha)}{2 \arctan \dfrac{(c_x - c_y)\sin \alpha \cos \alpha}{c_x \cos^2 \alpha + c_y \sin^2 \alpha}}$$

Proof of Lemma 5 (cf. Proof of Lemma 3)

Using $\psi \overline{x} = \begin{pmatrix} r & s \\ s & t \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_x^n \cos^2 \alpha + c_y^n \sin^2 \alpha & (c_x^n - c_y^n)\cos \alpha \sin \alpha \\ (c_x^n - c_y^n)\cos \alpha \sin \alpha & c_y^n \cos^2 \alpha + c_x^n \sin^2 \alpha \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ of Corollary 1

for the substitution of x and y resp. r and φ in the definition of log spirals:

$$R\sqrt{(a\cos \varphi + b\sin \varphi)^2 + (c\cos \varphi + d\sin \varphi)^2} = Re^{k\arctan \frac{c\cos\varphi + d\sin\varphi}{a\cos\varphi + b\sin\varphi}} \quad \text{leads to}$$

$$(c_x + c_y \cos^2\alpha)^2 + ((c_x - c_y)\sin\alpha \cos\alpha)^2 = e^{2k\arctan \frac{(c_x - c_y)\sin\alpha\cos\alpha}{c_x \cos^2\alpha + c_y \sin^2\alpha}} \quad \text{for } \varphi = 0$$

$$\Rightarrow 2k\arctan \frac{(c_x - c_y)\sin\alpha\cos}{c_x \cos^2\alpha + c_y \sin^2\alpha} = \ln(c_x^2\cos^2\alpha + c_y^2\sin^2\alpha)$$

Since for fixed α (e.g. $\tan\alpha \doteq 1$) different values of c_x and c_y form log spirals $Re^{k\varphi}$ with arbitrary k (different in ρ) k resp. 1/k can be represented in dependency of c_x and c_y:

$$k = \frac{\ln\left(\dfrac{c_x^2 + c_y^2}{2}\right)}{2\arctan \dfrac{c_x - c_y}{c_x + c_y}} \text{ resp. } \frac{1}{k} = \frac{\ln\left(\dfrac{c_x + c_y}{2}\right)}{2\arctan \dfrac{\sqrt{c_x} - \sqrt{c_y}}{\sqrt{c_x} + \sqrt{c_y}}} \quad \text{(for 1/k } c_x, c_y \text{ turn into } \sqrt{c_x} \text{ and } \sqrt{c_y} \text{ !)}$$

Now (A) the nominator resp. (B) the denominator of the above equations for k and 1/k can be equated to $\dfrac{-k\pi}{2(k+1)}$ resp. $\dfrac{-\pi}{2(k+1)}$ to calculate c_x and c_y.

(A) $\Rightarrow c^2{}_x + c^2{}_y = 2e^{\frac{-k\pi}{2(k+1)}}$ for k and $c_x + c_y = 2e^{\frac{-k^{-1}\pi}{2(k^{-1}+1)}}$ for k^{-1}

(B) $\Rightarrow c_x - c_y = -(c_x + c_y)\tan\dfrac{\pi}{4(k+1)}$ for k and $\sqrt{c_x} - \sqrt{c_y} = -\left(\sqrt{c_x} + \sqrt{c_y}\right)\tan\dfrac{\pi}{4(k^{-1}+1)}$ for k^{-1}

This leads to:

$$c_x^2 = e^{\frac{-k\pi}{2(k+1)}} \frac{\left(1 - \tan\frac{\pi}{4(k+1)}\right)^2}{\left(1 + \tan^2\frac{\pi}{4(k+1)}\right)} \quad \text{and} \quad c_y^2 = e^{\frac{-k\pi}{2(k+1)}} \frac{\left(1 + \tan\frac{\pi}{4(k+1)}\right)^2}{\left(1 + \tan^2\frac{\pi}{4(k+1)}\right)} \quad \text{for k}$$

$$\sqrt{c_x} = e^{\frac{-k^{-1}\pi}{2(k^{-1}+1)}} \frac{\left(1 - \tan\frac{\pi}{4(k^{-1}+1)}\right)^2}{\left(1 + \tan^2\frac{\pi}{4(k^{-1}+1)}\right)} \quad \text{and} \quad \sqrt{c_y} = e^{\frac{-k^{-1}\pi}{2(k^{-1}+1)}} \frac{\left(1 + \tan\frac{\pi}{4(k^{-1}+1)}\right)^2}{\left(1 + \tan^2\frac{\pi}{4(k^{-1}+1)}\right)} \quad \text{for } k^{-1}$$

(10)

Note: Now it obvious why k and 1/k is treated differently, because otherwise at least one of the resulting c_x or c_y can become greater than one in some cases.

The representation of an orbit ω^n by a continuos function of n (Theorem 1) allows to form the σ-Hull for an attractor A paramertized only by c_x and c_y and a fixed α (because in that case an arbitrary β leads to equal curved σ-Hulls according to Lemma 5). That is, to find an orbit generating transformation ω, where a maximal number of discrete boundary points of A lie on the orbit ω^n (for $n \in G$) forming the edge of a σ-Hull (for $n \in R$).

Now generating transformations ω_S of orbits can be calculated using the equivalent representation of a σ-Hull in R^2 if a σ-Hull is found, where at least four boundary points of A p_1, p_2, p_3, p_4 lay on one of the circumscribed spiral segments S with the fixed point f, forming the edge of this (best fitting) hull.

Thus, $\forall p_i \, i \in \{1,2,3,4\}$: $p_i \in \omega^n \neq p_i \in R_0(\pi/4)\omega^{-n} \neq p_i \in R_0(\pi/2)S(-1,1)\omega^n$ and ω^n forms the edge of the σ-Hull.

The c_x, c_y, α and β - for the log spiral equivalent (Theorem 1) - can be calculated from ω_S by using the representation $\omega = \psi R_0(-\beta)$ and $\psi = R_0(\alpha) S(c_x, c_y) R_0(-\alpha)$ (Lemma 1 and 2).

Optimizing the orbit generating transformations ω_S

$\omega_S = R_0(\alpha)S(c_x^n, c_y^n)R_0(-\alpha)R_0(n(-\beta))$ can now be optimized in the following way :

Since β can also be calculated as $\beta_k = \angle(p_k f R_0(\alpha_k) S(c^{-1}_x, c^{-1}_y) R_0(-\alpha_k)(p_{k+1}))$ and k=1,2,3
 $\Rightarrow$ a new $\beta = (\beta_1 + \beta_2 + \beta_3)/3$ can be obtained

Now a new c_x, c_y and α is recalculated using β_k to compute ψ_k:

$$\Rightarrow c_x = (c_{1x}+c_{2x}+c_{3x})/3, \, c_y = (c_{1y}+c_{2y}+c_{3y})/3 \text{ and } \alpha = (\alpha_1+\alpha_2+\alpha_3)/3$$

$$\text{where } c_{kx} = \frac{r_k + t_k + \sqrt{(r_k - t_k)^2 + 4s_k^2}}{2}, \; c_{ky} = \frac{r_k + t_k - \sqrt{(r_k - t_k)^2 + 4s_k^2}}{2} \text{ and } \alpha_k = \arctan\frac{c_{kx} - r_k}{s_k}$$

$$\text{for } \psi_k = \omega_S R_0(-\beta_k) = \begin{pmatrix} r_k & s_k \\ s_k & t_k \end{pmatrix} = R_0(\alpha_k) S(c_{kx}, c_{ky}) R_0(-\alpha_k) \text{ and k as in Lemma 2}$$

Reflection transformations (IFS-Codes) - Either c_x or c_y is negative

If two best fitting σ-Hulls are found which are formed by the log spirals

$$\omega_1^{2n}(p_1) = R_0(\alpha) S(c_x^{2n}, c_y^{2n}) R_0(-\alpha) R_0(2n(-\beta)) \text{ or}$$
$$\omega_2^{2n+1}(p'_1) = R_0(\alpha) S(c_x^{2n+1}, c_y^{2n+1}) R_0(-\alpha) R_0(-\beta(2n+1))$$

for $n \in N^0$ resp. $1/n \in N$ and p_1, $p'_1 \in R^2$, one affine reflective map can be derived :

$$\omega_{12} = R_0(\alpha)S(-c_x, c_y)R_0(-\alpha)R_0(-\beta) \text{ resp. } \omega_{12} = R_0(\alpha)S(c_x, -c_y)R_0(-\alpha)R_0(-\beta),$$

where $c_x < 0$ resp. $c_y < 0$, if $R_0(\gamma)\omega_1^n(p_1) = \omega_2^n(p'_1)$ resp. $R_0(-\gamma)\omega_1^n(p_1) = \omega_2^n(p'_1)$ for $\gamma \geq 0$.

2.4 How to calculate IFS-Codes

Deriving the IFS-Codes from boundary orbits

Form larger and larger curved σ-Hulls, where the c_x and c_y are computed according to Lemma 5 by using smaller and smaller k resp. 1/k (for $Re^{k\phi}$ in the log spiral equivalent of Theorem 1) for a fixed α *(Fig. 2)*.

Each time you get a best fitting hull (at least four boundary points of an attractor A lie on the edge of the σ-Hull) the ω_S - generating that orbit defined by these 4 points - is one IFS-Code of the IFS W_m for a subobject lying outside (i.e. having a part of its boundary in common with the object) as long as $\omega_S \neq \omega_{S'}$ *(Fig. 2)*. I.e. $W_{i+1} = W_i \cup \{\omega_S\}$ for i = {1, ..., m}, if $\omega_S \neq \omega_{S'}$ where $\omega_{S'} \in W_i$ and $\omega_S \equiv \omega_{S'}$, if $\omega_{S'} = \omega_S\omega_{S''}$ where $\omega_{S''} \in W_i$
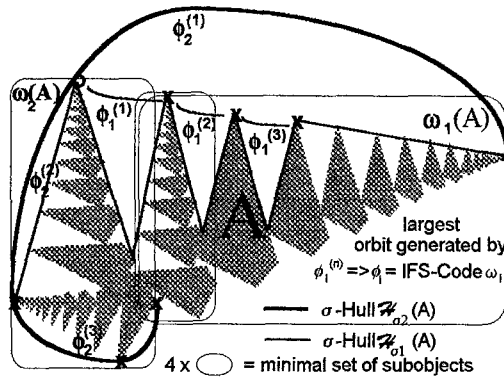

Figure 2. IFS-Code calculation by orbit generating transformations

Besides, this extremely simplifies the classification process (cf. introduction) because larger orbits are found first and the smaller ones have not to be put into the classes any more. Using $\omega^n = R_0(\alpha)S(c_x{}^n, c_y{}^n)R_0(-\alpha)R_0(n(-\beta))$, several special cases can be apriori excluded and thus unnecessary classes are not formed:

- Multiples of other orbits are detected if $S(c'_x, c'_y) = S(c_x{}^n, c_y{}^n)$ n $\in$ G resp. if $\alpha_S = \alpha_{S'}$.
- The orbit generating transformations ω_S of the largest orbits of one class is that ω'_S with the largest β.
- Orbits which can be affinely mapped (by inter orbit transformations) onto eachother will now be detected by those ω_S having a multiple in β - without using an affine invariant representation[7,8].

Thus in the classification process it is not longer necessary to use classes of orbits but classes of the orbit generating and inter orbit transformations represented by their log spiral equivalent (of Theorem 1)

Deriving the IFS-Codes from non boundary orbits

Find all ω_{kj} that for at least one of the spirals S_{1i} (which are formed by each best fitting σ-Hull of A for the IFS-Codes ω_{1i}) circumscribed to $A \backslash \omega^{-1}{}_{kj} (\bigcup \omega_{k-1i} A)$) holds $\omega_{kj}(S_{1i}) = S_{ki}$ and S_{ki} a circumscribed spiral to $A \backslash \bigcup^i \omega_{k-1i}(A)$ and ω_{kj} the IFS-Codes for an inner subobject (i.e. having to less boundary points in common with the object).

Repeat this process until $A \setminus \overset{i}{\bigcup} \omega_{ki}(A) = \varnothing$, where $\{\omega_{ki}\} = \{\omega_{k-1i}\} \cup \{\omega_{kj}\}$ and $k = \{2,...,n\}$.

Note: Thus the IFS-Codes of attractors with overlapping subobjects can also be calculated.

3   Conclusion

3.1   Summary

In this work is shown how the boundary of a discrete self affine object can be calculated. This is the proposition to form orbits of boundary points of an object. Subsequently the orbits have to be classified to calculate the IFS-Codes (affine maps) of a minimal IFS as generating transformations of the largest of these orbits in each class.

The boundary orbits are found by using a generalized convex hull - the σ-Hull. This hull is formed by the intersection of generalized spiral disk segments (parts of the plane bounded by 3 log spirals equal in curvature proportions and fixed points) instead of half planes used for the convex hull. To reduce the calculation costs, an equivalent representation of spirals (in the polar plane) is defined by powers of affine transformations (in the x/y-plane).

This representation has the additional advantage, that - by the used affine transformations - orbits can easily be calculated for the subsequent classification. Besides, the affine equivalent to log spirals extremely reduces the expense for comparison and differentiation of the orbits within the classification process.

Further more generating transformations for non boundary orbits can be detected after removing the subobjects lying outside which have been already calculated. Thus the IFS-Codes for subobjects having no boundary in common with the object can be computed, even if the subobjects are overlapping.

3.2   Future Work

Implementation

This is a theoretical work, which forms the basis for the IFS-Code calculation and classification resp. overcomes the difficulties of their calculation existing so far. Therefore an implementation will be made together with the elaboration of the classification part which has to be completed to solve the inverse IFS-Problem as a whole.

But the prototype developed for the last work[9] already shows, that in principle this problem is solvable by using a less general hull similar to the σ-Hull and by mapping orbits of boundary points onto eachother.

Besides, the experience made by the implementation of this prototype showed the need to have a more general tool (the σ-Hull) to calculate and classify the orbits as basis for the computation of a minimal IFS.

Analysing IFS-attractors under the aspects of the σ-Hull

So far the affine equivalent of log spirals (the basis of the σ-Hull) has primarily be used to calculate the boundary of an IFS-object. But it can also be an instrument to categorize IFS-objects according to the intrinsic properties, such as structure, location and form of the object parts (subobjects) resp. orbits.

Therefore the correlation between orbits and the parameters of the σ-Hull such as scaling factors (and powers of them) - determining the curvature of the orbits - and the

two angles - one determining the distance of orbit points and the other determining the contraction of an orbit - have to be studied in further detail.

The gained knowledge will also improve the IFS-calculation and contribute to a better adaptation of the σ-Hull.

Can the classification process be more improved by the σ-Hull ?

Some implications of the parameters of the σ-Hull for the classification have been already analysed in this paper, but there should be more implicit properties of the σ-Hull which can be used to classify the orbits.

Besides the choice of the order - not only in dependency of a decreasing function of the curvature proportion - by which the σ-Hulls are formed can influence the discrimination process of the detected orbits to be classified.

Last but not least the affine equivalent to log spirals can be used - within the classification - to decide if one matrix can be divided by another resp. for the comparison of matrices with respect to their fixed points.

Designing a more adequate σ-Hull

Based on the present knowledge about orbits of boundary points of IFS-objects, the basic element of the σ-Hull was designed as the intersection of 3 spiral disks equal in curvature proportion and fixed point. But in praxis it may turn out that more or less log spirals with different curvature and fixed point will do a better job.

This variations also will change the size of the basic element, what will influence the number and accuracy of orbits found on the boundary. That is, roughly speaking how "tight" σ-Hulls can be formed.

So far only two types of basic elements have been investigated. An intersection of two log spiral disks having the same curvature and the same fixed point. But - though simpler to handle - this leads to basic elements infinite in size, which are not sufficient to circumscribe an IFS-object in several cases.

Another basic element was formed by two log spiral disks identically curved in opposite direction and having the same tangent in their two tangent points. This leads to more consistent (with respect to the definition of hulls) mathematical model of the σ-Hull, but this basic element is convex and using it's complement again leads to less "tight" σ-Hulls, which prevent that the fixed points of orbits to lay on the boundary.

Generalization of the σ-Hull

In general each discrete object can be represented by an IFS as long as sufficient many IFS-Codes are used (in the extreme case one IFS-Code per pixel !). Decomposing an object in adequate parts will reduce the number of IFS-Codes[10]. So the question is not to design more and more complex hulls (bounded by highly non linear curves[11]), but to use the σ-Hull to find an minimal number of orbits fitting best into the calculated boundary points. Two possible solutions are outlined below:

One to enlarge the pixel size by an appropriate scale and to fit the orbits into less boundary points. Another to make the pixels smaller - so that each point becomes a boundary point and to find all different orbits within this unconnected point set. The second approach seems to be more complicated (complexity in the point relation), but it solves the encoding problem in one step, because there are no subobjects lying inside, anymore.

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